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Semiclassical states for the nonlinear Schrödinger equation on saddle points of the potential via variational methods ☆

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Abstract

In this paper we study semiclassical states for the problem

$$-\varepsilon^2 \Delta u + V(x)u = f(u) \quad \text{in } \mathbb{N},$$

where $f(u)$ is a superlinear nonlinear term. Under our hypotheses on f a Lyapunov–Schmidt reduction is not possible. We use variational methods to prove the existence of spikes around saddle points of the potential $V(x)$.

Keywords: Nonlinear Schrödinger equation; Semiclassical states; Variational methods

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1. Introduction

Our starting point is the equation of the standing waves for the Nonlinear Schrödinger Equation

$$-\varepsilon^2 \Delta u + V(x)u = f(u) \quad \text{in } \mathbb{R}^N. \tag{1}$$

Here $u \in H^1(\mathbb{R}^N)$, $N \geqslant 2$, $V(x)$ is a positive potential and f is a nonlinear term. This problem has been largely studied in the literature and it is not possible to give here a complete bibliography.

The existence of solutions for (1) has been treated in [8,31] for constant potentials and [5,6,16,30] in more general cases. An interesting issue concerning (1) is the existence of semiclassical states, which implies the study of (1) for small $\varepsilon > 0$. From the point of view of Physics, semiclassical states describe a kind of transition from Quantum Mechanics to Newtonian Mechanics. In this framework one is interested not only in existence of solutions but also in their asymptotic behavior as $\varepsilon \to 0$. Typically, solutions tend to concentrate around critical points of V: such solutions are called *spikes*.

The first result in this direction was given by Floer and Weinstein in [19], where the case $N = 1$ and $f(u) = u^3$ is considered. Later, Oh generalized this result to higher values of N and $f(u) = u^p$, $1 < p < \frac{N+2}{N-2}$, see [28,29]. In those papers existence of *spikes* around any non-degenerate critical point x_0 of $V(x)$ is proved. Roughly speaking, a spike is a solution u_ε such that

$$u_\varepsilon \sim U\left(\frac{x - x_0}{\varepsilon}\right), \quad \text{as } \varepsilon \to 0,$$

where U is a ground state solution of the limit problem

$$-\Delta U + V(x_0)U = f(U). \tag{2}$$

Let us point out here that not any critical point of $V(x)$ generates a spike around it: for instance, it has been proved in [17,18] that (1) has no nontrivial solution if $V(x)$ is decreasing along a direction (and different from constant). However, [1,25] extended the previous result to some possibly degenerate critical points of V.

All those results [1,19,25,28,29] use the following non-degeneracy condition for (2):

(ND) the vector space of solutions of $-\Delta w + V(x_0)w = f'(U)w$ is generated by $\{\partial_{x_i} U, i = 1, \ldots, N\}$.

This property is essential in their approach since they use a Lyapunov–Schmidt reduction which is based on the study of the linearized problem. The argument of the proof of (ND) (see for instance [2, Chapter 4]) needs a non-existence result for ODE's that has been proved only for specific types of nonlinearities, like powers (see [23]).

A first attempt to generalize such result without assuming (ND) was given in [13] (see also [20]), which was later improved by [14,15]. Here the procedure is completely different and uses a variational approach applied to a truncated problem. In those papers the following hypotheses are made on f:

(f0) $f : [0, +\infty) \to [0, +\infty)$ is a C^1 function;
(f1) $f(s) = o(s)$ as $s \sim 0$;
(f2) $\lim_{s\to+\infty} \frac{f(s)}{s^p} = 0$ for some $p \in (1, \frac{N+2}{N-2})$ if $N \geqslant 3$, or just $p > 1$ if $N = 2$;
(f3) there exists $\mu > 2$ such that, for every $s > 0$,

$$0 < \mu F(s) \leqslant sf(s),$$

where $F(s) = \int_0^s f(t)\,dt$;
(f4) the map $t \mapsto \frac{f(t)}{t}$ is non-decreasing.

Conditions (f1)–(f2) imply that f is superlinear and sub-critical, and are quite natural in this framework. Condition (f3) is the so-called Ambrosetti–Rabinowitz condition, which has been imposed many times in order to deal with superlinear problems. Finally, condition (f4) is suitable for using a Nehari manifold approach.

Under those conditions, [15] shows the existence of spikes around certain critical points of $V(x)$. Roughly speaking, the critical points considered are those that can be found through a local min–max approach; this is a very general assumption and includes of course any non-degenerate critical point.

Recently, some papers have tried to eliminate some of the conditions (f3)–(f4) or to substitute them with other assumptions. For instance, in [12,22] condition (f4) is removed (moreover, [22] deals also with asymptotically linear problems, where (f3) is replaced with another condition). In [4,9,10] both conditions (f3) and (f4) are eliminated and the authors assume the minimal hypotheses under which one can prove the existence of solution for (2) (those of [8]). However, in [4,9,10,12,22] only the case of local minima of $V(x)$ is considered.

The goal of this paper is to prove existence of spikes around saddle points or maxima of $V(x)$ without assumption (f4). Our approach is reminiscent of [15]: basically, we define a conveniently modified energy functional and try to prove existence of solution by variational methods. The main difference with respect to [15] is that, since (f4) is not assumed, the Nehari manifold technique is not applicable here. So, we need to construct a different min–max argument, which involves suitable deformations of certain cones in $H^1(\mathbb{R}^N)$. This approach seems very natural but has not been used before in the related literature. As a second novelty, a classical property of the Brouwer degree regarding the existence of connected sets of solutions reveals crucial to estimate the critical values (see [24,27]). Indeed, this property allows us to relate our min–max value to another min–max value with the constraint of having center of mass equal to 0 (see Section 3 for a more detailed exposition).

Finally, once a solution is obtained, asymptotic estimates are needed in order to prove that the solution of the modified problem solves (1).

We assume that $V : \mathbb{R}^N \to \mathbb{R}$ is a function satisfying the following boundedness condition:

(V0) $0 < \alpha_1 \leqslant V(x) \leqslant \alpha_2$ for all $x \in \mathbb{R}^N$.

Moreover, with respect to the critical point 0, we assume one of the following conditions:

(V1) $V(0) = 1$, V is C^1 in a neighborhood of 0 and 0 is an isolated local maximum of V;
(V2) $V(0) = 1$, V is C^2 in a neighborhood of 0 and 0 is a non-degenerate saddle critical point of V;

(V3) $V(0) = 1$, V is C^{N-1} in a neighborhood of 0, 0 is an isolated critical point of $V(x)$ and there exists a vector space E such that:
(a) $V|_E$ has a local maximum at 0,
(b) $V|_{E^\perp}$ has a local minimum at 0.

Our assumptions on the critical points of V are not as general as in [15], but still include non-degenerate cases, as well as isolated maxima and many degenerate cases.

Our main theorem is the following.

Theorem 1.1. *Assume that f satisfies hypotheses* (f0), (f1), (f2), (f3), *and that V satisfies* (V0) *and one of* (V1), (V2) *or* (V3). *Then there exists $\varepsilon_0 > 0$ such that* (1) *admits a positive solution u_ε for $\varepsilon \in (0, \varepsilon_0)$. Moreover, there exists $\{y_\varepsilon\} \subset \mathbb{R}^N$ such that, as $\varepsilon \to 0$, $\varepsilon y_\varepsilon \to 0$ and*

$$u_\varepsilon\big(\varepsilon(\cdot + y_\varepsilon)\big) \to U \quad \text{in } H^1\big(\mathbb{R}^N\big),$$

where U is a ground state solution for

$$-\Delta U + U = f(U).$$

This result can be compared with [9,15] as follows. In [15] more general critical points of the potential $V(x)$ are considered, but condition (f4) is assumed. On the other hand, the hypotheses on f of [9] are less restrictive than ours, but [9] considers only local minima of V.

The remaining of the paper is organized as follows. In Section 2 we will define the truncation of the problem that will be used throughout the paper. Moreover we will give some preliminary results, most of them well known, related to the autonomous problem. The min–max argument is exposed in Section 3, where we get the existence of a solution for the truncated problem. For the sake of clarity, the main estimate needed in our argument, Proposition 3.4, will be proved in Section 4. In Section 5 some asymptotic estimates on the solutions will be given: in particular we will show that the solutions of the truncated problem actually solve our original problem. A couple of technical results which are needed in our arguments are proved in a final Appendix A, where also some extensions of our result are briefly commented.

After the completion of this paper we learned that a very general result on this topic has been achieved in a recent preprint [11]. The arguments of the proofs there are quite different from ours.

2. Preliminaries

In this section we give some preliminary definitions and results that will be used in our arguments. First, we define a certain truncation of $f(u)$ and establish the basic properties of the related problem. After that, we will address the study of certain limit problems that will appear naturally in later proofs.

Let us first fix some notations. In $\mathbb{R}^N$, $B(x, R)$ will denote the usual Euclidean ball centered at $x \in \mathbb{R}^N$ and with radius $R > 0$. Given any set $\Lambda \subset \mathbb{R}^N$, its complement is denoted by Λ^c. Moreover, for any $\varepsilon > 0$, we write

$$\Lambda^\varepsilon = \varepsilon^{-1}\Lambda = \big\{x \in \mathbb{R}^N\colon\ \varepsilon x \in \Lambda\big\}.$$

In what follows we will denote by $\|\cdot\|$ the usual norm of $H^1(\mathbb{R}^N)$: other norms, like Lebesgue norms, will be indicated with a subscript. If nothing is specified, strong and weak convergence of sequences of functions are assumed in the space $H^1(\mathbb{R}^N)$.

In our estimates, we will frequently denote by $C > 0$, $c > 0$ fixed constants, that may change from line to line, but are always independent of the variable under consideration. We also use the notations $O(1), o(1), O(\varepsilon), o(\varepsilon)$ to describe the asymptotic behaviors of quantities in a standard way.

2.1. The truncated problem

By making the change of variable $x \mapsto \varepsilon x$, problem (1) becomes

$$-\Delta u + V(\varepsilon x)u = f(u) \quad \text{in } \mathbb{R}^N. \tag{3}$$

Throughout the paper we will extend $f(u)$ to be equal to 0 for negative values of u. Observe that, by the maximum principle, any nontrivial solution of (3) will be positive, so that we come back to our original problem.

It is well known that solutions of (3) correspond to critical points of the functional $I_\varepsilon : H^1(\mathbb{R}^N) \to \mathbb{R}$,

$$I_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)u^2 - \int_{\mathbb{R}^N} F(u).$$

However, we will not deal with (3) and I_ε directly. First, we use a convenient truncation of the nonlinear term $f(u)$, in the line of [13–15,22]. The idea is to localize the problem around 0, so that the energy functional becomes coercive far from the origin.

Let us define

$$\tilde{f}(s) = \begin{cases} \min\{f(s), as\}, & s \geqslant 0, \\ 0, & s < 0, \end{cases}$$

with

$$0 < a < \left(1 - \frac{2}{\mu}\right)\alpha_1. \tag{4}$$

We also define the primitive $\tilde{F}(s) = \int_0^s \tilde{f}(t)\,dt$.

In the following we will consider the balls $B_i = B(0, R_i) \subset \mathbb{R}^N$ $(i = 0, \dots, 4)$ with $R_i < R_{i+1}$ for $i = 0, 1, 2, 3$, where R_i are small positive constants to be determined. For technical reasons, we will choose R_1 such that:

$$\forall x \in \partial B_1 \text{ with } V(x) = 1, \quad \partial_\tau V(x) \neq 0, \quad \text{where } \tau \text{ is tangent to } \partial B_1 \text{ at } x. \tag{5}$$

In cases (V1) and (V2) it is clear that such a choice is possible. In case (V3) this is also true thanks to the Sard Lemma, see Proposition A.1 in Appendix A. This is the unique point where the C^{N-1} regularity of V is needed in case (V3).

Next we define $\chi : \mathbb{R}^N \to \mathbb{R}$,

$$\chi(x) = \begin{cases} 1, & x \in B_1, \\ \frac{R_2 - |x|}{R_2 - R_1}, & x \in B_2 \setminus B_1, \\ 0, & x \in B_2^c, \end{cases} \tag{6}$$

and then

$$g(x,s) = \chi(x) f(s) + \bigl(1 - \chi(x)\bigr) \tilde{f}(s),$$
$$G(x,s) = \int_0^s g(x,t)\,dt = \chi(x) F(s) + \bigl(1 - \chi(x)\bigr) \tilde{F}(s).$$

We denote with subscripts the dilation of the previous functions. Being more specific, we denote

$$\chi_\varepsilon(x) = \chi(\varepsilon x),$$

and

$$g_\varepsilon(x,s) = g(\varepsilon x, s) = \chi_\varepsilon(x) f(s) + \bigl(1 - \chi_\varepsilon(x)\bigr) \tilde{f}(s).$$

So, in this section we consider the truncated problem

$$-\Delta u + V(\varepsilon x) u = g_\varepsilon(x,u) \quad \text{in } \mathbb{R}^N. \tag{7}$$

As mentioned above, we will find solutions of (7) as critical points of the associated energy functional $\tilde{I}_\varepsilon : H^1(\mathbb{R}^N) \to \mathbb{R}$, which is defined as

$$\tilde{I}_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(\varepsilon x) u^2 - \int_{\mathbb{R}^N} G_\varepsilon(x,u),$$

with

$$G_\varepsilon(x,s) = \int_0^s g_\varepsilon(x,t)\,dt = \chi_\varepsilon(x) F(s) + \bigl(1 - \chi_\varepsilon(x)\bigr) \tilde{F}(s).$$

In the next lemma we collect some properties of the functions defined above that will be of use in our reasonings.

Lemma 2.1. *There hold*:

($\tilde{\text{f}}$1) $\tilde{F}(s) \leqslant \min\{\frac{1}{2} a s^2, F(s)\}$;
($\tilde{\text{f}}$2) *there exists* $r > 0$ *such that* $\tilde{f}(s) = f(s)$ *for* $s \in (0, r)$;

(g1) *$G_\varepsilon(x,s) \leqslant F(s)$ for all $(x,s) \in \mathbb{R}^N \times \mathbb{R}$;*
(g2) *$g_\varepsilon(x,s) = f(s)$ if $|s| < r$ or $x \in B_1^\varepsilon$;*
(fg) *for any $\delta > 0$ there exists $C_\delta > 0$ such that*

$$\left|f(s)\right| \leqslant \delta|s| + C_\delta|s|^p,$$

and the same assertion also holds for $\tilde{f}(s)$, $g_\varepsilon(x,s)$.

Proof. Properties ($\tilde{\text{f}}$1), ($\tilde{\text{f}}$2), (g1), (g2) follow immediately from the definitions of $\tilde{f}$ and g_ε. Finally, property (fg) follows from the assumptions (f1) and (f2). □

Proposition 2.2. *For every $\varepsilon > 0$, the functional $\tilde{I}_\varepsilon$ satisfies the Palais–Smale condition.*

The proof of this result is basically identical to the proof of [13, Lemma 1.1]. We reproduce it here for the sake of completeness.

Proof. Let $\{u_n\}$ be a (PS) sequence for $\tilde{I}_\varepsilon$. There hold

$$\tilde{I}_\varepsilon(u_n) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)u_n^2 - \int_{\mathbb{R}^N} G_\varepsilon(x,u_n) \to c$$

and

$$\tilde{I}_\varepsilon'(u_n)[u_n] = \int_{\mathbb{R}^N} |\nabla u_n|^2 + \int_{\mathbb{R}^N} V(\varepsilon x)u_n^2 - \int_{\mathbb{R}^N} g_\varepsilon(x,u_n)u_n = o\big(\|u_n\|\big).$$

By (4) we have

$$\begin{aligned}
\mu\tilde{I}_\varepsilon(u_n) - \tilde{I}_\varepsilon'(u_n)[u_n] &= \left(\frac{\mu}{2}-1\right)\int_{\mathbb{R}^N}\big(|\nabla u_n|^2 + V(\varepsilon x)u_n^2\big) - \int_{\mathbb{R}^N}\chi_\varepsilon(x)\big(\mu F(u_n) - f(u_n)u_n\big)\\
&\quad - \int_{\mathbb{R}^N}\big(1-\chi_\varepsilon(x)\big)\big(\mu\tilde{F}(u_n) - \tilde{f}(u_n)u_n\big)\\
&\geqslant \left(\frac{\mu}{2}-1\right)\int_{\mathbb{R}^N}\big(|\nabla u_n|^2 + V(\varepsilon x)u_n^2\big) - \frac{\mu}{2}a\int_{\mathbb{R}^N} u_n^2\\
&\geqslant c\|u_n\|^2.
\end{aligned}$$

Then $\{u_n\}$ is bounded and hence $u_n \rightharpoonup u$ up to a subsequence. Now we show that this convergence is strong. It is sufficient to prove that for every $\delta > 0$ there exists $R > 0$ such that

$$\limsup_n \|u_n\|_{H^1(B(0,R)^c)} < \delta.$$

We take $R > 0$ such that $B_2^\varepsilon \subset B(0, R/2)$. Let ϕ_R be a cut-off function such that $\phi_R = 0$ in $B(0, R/2)$, $\phi_R = 1$ in $B(0, R)^c$, $0 \leqslant \phi_R \leqslant 1$ and $|\nabla \phi_R| \leqslant C/R$. Then

$$\tilde{I}'_\varepsilon(u_n)[\phi_R u_n] = \int\limits_{\mathbb{R}^N} \nabla u_n \cdot \nabla(\phi_R u_n) + \int\limits_{\mathbb{R}^N} V(\varepsilon x) u_n^2 \phi_R - \int\limits_{\mathbb{R}^N} g_\varepsilon(x, u_n)\phi_R u_n = o_n(1)$$

since $\{u_n\}$ is bounded. Therefore

$$\begin{aligned}\int\limits_{\mathbb{R}^N} \big(|\nabla u_n|^2 + V(\varepsilon x) u_n^2\big)\phi_R &= \int\limits_{\mathbb{R}^N} \tilde{f}(u_n)\phi_R u_n - \int\limits_{\mathbb{R}^N} u_n \nabla u_n \cdot \nabla \phi_R + o_n(1) \\ &\leqslant a \int\limits_{\mathbb{R}^N} u_n^2 + \frac{C}{R} + o_n(1)\end{aligned}$$

and so

$$\|u_n\|^2_{H^1(B(0,R)^c)} \leqslant C/R + o_n(1). \qquad \square$$

2.2. *The limit problems*

Let us start by studying the limit problem

$$-\Delta u + ku = f(u) \tag{$\mathcal{LP}_k$}$$

for some $k > 0$. The associated energy functional $\Phi_k : H^1(\mathbb{R}^N) \to \mathbb{R}$ is defined as

$$\Phi_k(u) = \frac{1}{2} \int\limits_{\mathbb{R}^N} |\nabla u|^2 + \frac{k}{2} \int\limits_{\mathbb{R}^N} u^2 - \int\limits_{\mathbb{R}^N} F(u). \tag{8}$$

Observe that any solution of $(\mathcal{LP}_k)$ must be positive by the maximum principle (recall that $f(u)$ is extended to be 0 for negative values of u).

Problem $(\mathcal{LP}_k)$ can be attacked by using the Mountain Pass Theorem in a radially symmetric framework, see [3,7,8]. Indeed, let us define

$$m_k = \inf_{\gamma \in \Gamma_k} \max_{t \in [0,1]} \Phi_k\big(\gamma(t)\big), \tag{9}$$

with $\Gamma_k = \{\gamma \in C([0,1], H^1(\mathbb{R}^N)):\ \gamma(0) = 0,\ \Phi_k(\gamma(1)) < 0\}$. It can be proved that m_k is a critical value of Φ_k, that is, there exists a solution $U \in H^1(\mathbb{R}^N)$ of $(\mathcal{LP}_k)$ such that $\Phi_k(U) = m_k$. Moreover, it is known that U is a ground state solution or, in other words, it is the solution with minimal energy, see [21].

However, without some additional hypotheses on f, it is not known whether that solution is unique or not. Every non-negative solution U of $(\mathcal{LP}_k)$ satisfies the following properties (see [8,31]):

(i) $U(x) > 0$, U is C^∞ and radially symmetric (up to a translation);
(ii) $U(r)$ is decreasing in $r = |x|$ and converges to zero exponentially as $r \to +\infty$;
(iii) the Pohozaev identity

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla U|^2 + k\frac{N}{2}\int_{\mathbb{R}^N} U^2 = N\int_{\mathbb{R}^N} F(U)$$

holds.

The following lemma basically states that the infimum in (9) is actually a minimum.

Lemma 2.3. *(See [21, Lemma 2.1].) Let $U \in H^1(\mathbb{R}^N)$ a ground state solution of $(\mathcal{LP}_k)$. Then, there exists $\gamma \in \Gamma_k$ such that $U \in \gamma([0,1])$ and*

$$\max_{t\in[0,1]} \Phi_k\big(\gamma(t)\big) = m_k.$$

Let us briefly describe the construction given in [21]. Given $t > 0$, we denote

$$U_t = U\left(\frac{\cdot}{t}\right), \quad t > 0.$$

For $N \geqslant 3$, the curve γ is constructed by simply dilating the space variable:

$$\gamma(t) = \begin{cases} U_t & \text{if } t \in (0,\theta], \\ 0 & \text{if } t = 0. \end{cases}$$

For $N = 2$ the construction of γ is given as a concatenation of the following three curves:

$$\begin{cases} tU_{\theta_0} & \text{if } t \in [0,1], \\ U_\theta & \text{if } \theta \in [\theta_0,\theta_1], \\ tU_{\theta_1} & \text{if } t \in [1,\bar{t}], \end{cases}$$

with suitable $\theta_0 \in (0,1)$ and $\theta_1, \bar{t} > 1$. Observe that in both cases γ is defined in a closed interval: a suitable re-parametrization of it gives us the desired curve.

This curve will be of use for the construction of our min–max scheme.

The following lemma studies the dependence of the critical level on k.

Lemma 2.4. *The map $m : (0,+\infty) \to (0,+\infty)$,*

$$m(k) = m_k$$

is strictly increasing and continuous.

Proof. Let us first show that m is strictly increasing. Take $k_1, k_2 > 0$ with $k_1 < k_2$ and $\gamma \in \Gamma_{k_2}$ given by Lemma 2.3. Observe that clearly $\gamma \in \Gamma_{k_1}$ and

$$m_{k_1} \leqslant \max_{t\in[0,1]} \Phi_{k_1}\big(\gamma(t)\big) < \max_{t\in[0,1]} \Phi_{k_2}\big(\gamma(t)\big) = m_{k_2}.$$

We now prove the continuity of m. Take $\{k_j\}$ a sequence of positive real numbers that converges to $k > 0$. As above, take $\gamma \in \Gamma_k$ given by Lemma 2.3. For sufficiently large j, $\gamma \in \Gamma_{k_j}$ and

$$m_{k_j} \leqslant \max_{t\in[0,1]} \Phi_{k_j}\big(\gamma(t)\big) \to \max_{t\in[0,1]} \Phi_k\big(\gamma(t)\big) = m_k.$$

Then

$$\limsup_j m_{k_j} \leqslant m_k.$$

We now prove a reversed inequality. For every $j \in \mathbb{N}$, we consider U_j a radially symmetric least energy solution of

$$-\Delta u + k_j u = f(u). \tag{$\mathcal{LP}_{k_j}$}$$

The sequence $\{U_j\}$ is bounded in $H^1(\mathbb{R}^N)$-norm. Indeed, since

$$\frac{1}{2}\int_{\mathbb{R}^N} |\nabla U_j|^2 + \frac{k_j}{2}\int_{\mathbb{R}^N} U_j^2 - \int_{\mathbb{R}^N} F(U_j) = m_{k_j} = O(1)$$

and

$$\int_{\mathbb{R}^N} |\nabla U_j|^2 + k_j\int_{\mathbb{R}^N} U_j^2 - \int_{\mathbb{R}^N} f(U_j)U_j = 0,$$

then, by (f3), we obtain

$$\begin{aligned}\mu m_{k_j} &= \left(\frac{\mu}{2} - 1\right)\int_{\mathbb{R}^N} |\nabla U_j|^2 + k_j\left(\frac{\mu}{2} - 1\right)\int_{\mathbb{R}^N} U_j^2 - \int_{\mathbb{R}^N} \mu F(U_j) - f(U_j)U_j \\ &\geqslant c\|U_j\|^2.\end{aligned}$$

Therefore $U_j \rightharpoonup U$ in $H_r^1(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N)\colon u \text{ is radially symmetric}\}$. By the compact embedding of $H_r^1(\mathbb{R}^N)$ into $L^{p+1}(\mathbb{R}^N)$ (see [31]), we get that $U_j \to U$ in $L^{p+1}(\mathbb{R}^N)$.

Since

$$\frac{1}{2}\int_{\mathbb{R}^N} |\nabla U_j|^2 + \frac{k_j}{2}\int_{\mathbb{R}^N} U_j^2 = m_{k_j} + \int_{\mathbb{R}^N} F(U_j) \geqslant c > 0$$

and, by (fg), fixed $\delta > 0$ small enough,

$$0 < c \leqslant \int_{\mathbb{R}^N} |\nabla U_j|^2 + (k_j - \delta) \int_{\mathbb{R}^N} U_j^2 \leqslant C \int_{\mathbb{R}^N} |U_j|^{p+1},$$

so that $U \neq 0$.

Observe that U is a positive radially symmetric solution of the problem

$$-\Delta U + kU = f(U).$$

Moreover, using the strong convergence in $L^{p+1}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} F(U_j) \to \int_{\mathbb{R}^N} F(U).$$

By the lower semicontinuity of the $H^1(\mathbb{R}^N)$ norm, we conclude

$$\liminf_{j\to+\infty} m_{k_j} = \liminf_{j\to+\infty} \Phi_{k_j}(U_j) \geqslant \Phi_k(U) \geqslant m_k. \qquad \square$$

We finish the section with a couple of definitions that will be of use later. For simplicity, when we restrict ourselves to the case $k = 1$, we denote

$$\Phi = \Phi_1, \qquad m = m_1. \tag{10}$$

Let us define

$$\mathcal{S} = \{u \in H^1(\mathbb{R}^N):\ \Phi(u) = m,\ \Phi'(u) = 0\}. \tag{11}$$

In other words, $\mathcal{S}$ denotes the set of positive ground state solutions of the problem

$$-\Delta u + u = f(u).$$

Moreover, given any $y \in \mathbb{R}^N$, we define the energy functional $J_y : H^1(\mathbb{R}^N) \to \mathbb{R}$,

$$J_y(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{V(y)}{2} \int_{\mathbb{R}^N} u^2 - \int_{\mathbb{R}^N} G(y, u). \tag{12}$$

Obviously, the critical points of J_y are solutions of the problem

$$-\Delta u + V(y)u = g(y, u).$$

3. The min–max argument

In this section we develop the min–max argument that will provide the existence of a solution. In the proof, we will show that the min–max value m_ε converges to m as $\varepsilon \to 0$ (see (10) for the definition of m). With this, the existence result for the truncated problem follows quite easily.

In the proof of the convergence $m_\varepsilon \to m$, the hardest part is the estimate from below. This will be accomplished by showing that $m_\varepsilon \geqslant b_\varepsilon$, where b_ε is another min–max value to be defined. A key ingredient for that comparison is a classical property of the Brouwer degree concerning existence of connected sets of solutions. The fact that $\liminf_{\varepsilon\to 0} b_\varepsilon \geqslant m$ will be proved in next section.

First of all, let us observe that, under our hypotheses on V, there exists a vector space E such that:

(a) $V|_E$ has a strict local maximum at 0;
(b) $V|_{E^\perp}$ has a strict local minimum at 0.

Indeed, in case (V1), $E = \mathbb{R}^N$, whereas, in case (V2), E is the space spanned by eigenvectors associated to negative eigenvalues of $D^2V(0)$.

We begin by defining the topological cone

$$\mathcal{C}_\varepsilon = \bigl\{\gamma_t(\cdot - \xi)\colon\ t \in [0,1],\ \xi \in \overline{B_0^\varepsilon} \cap E\bigr\}. \tag{13}$$

Here $\gamma_t = \gamma(t)$ is the curve given in Lemma 2.3 for $k = 1$ and U a radially symmetric ground state. Observe that $\gamma(0) = 0$ is the vertex of the cone. Let us define a family of deformations of $\mathcal{C}_\varepsilon$

$$\Gamma_\varepsilon = \bigl\{\eta : \mathcal{C}_\varepsilon \to H^1\bigl(\mathbb{R}^N\bigr) \text{ homeomorphism: } \eta(u) = u\ \forall u \in \partial\mathcal{C}_\varepsilon\bigr\},$$

where $\partial\mathcal{C}_\varepsilon$ is the topological boundary of $\mathcal{C}_\varepsilon$. Recall that $m = m_1$ is the ground state energy level of the problem $-\Delta u + u = f(u)$, see (9).

We define the min–max level

$$m_\varepsilon = \inf_{\eta\in\Gamma_\varepsilon} \max_{u\in\mathcal{C}_\varepsilon} \tilde{I}_\varepsilon\bigl(\eta(u)\bigr).$$

Proposition 3.1. *There exist $\varepsilon_0 > 0$, $\delta > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$*

$$\tilde{I}_\varepsilon|_{\partial\mathcal{C}_\varepsilon} \leqslant m - \delta.$$

Proof. It suffices to show that

$$\tilde{I}_\varepsilon\bigl(\gamma_1(\cdot - \xi)\bigr) < 0 \quad \forall \xi \in B_0^\varepsilon \cap E \tag{14}$$

and

$$\tilde{I}_\varepsilon\bigl(\gamma_t(\cdot - \xi)\bigr) < m - \delta \quad \forall \xi \in \partial B_0^\varepsilon \cap E,\ t \in [0,1]. \tag{15}$$

Let us denote $\hat{U} = \gamma_1(\cdot - \xi)$ for some $\xi \in B_0^\varepsilon \cap E$. Then,

$$\tilde{I}_\varepsilon(\hat{U}) \leqslant \int\limits_{B_1^\varepsilon} \left(\frac{1}{2}|\nabla \hat{U}|^2 + \frac{1}{2}V(\varepsilon x)\hat{U}^2 - F(\hat{U})\right) + \int\limits_{(B_1^\varepsilon)^c} \left(\frac{1}{2}|\nabla \hat{U}|^2 + \frac{1}{2}V(\varepsilon x)\hat{U}^2\right).$$

By the exponential decay of $\hat{U}$, we get

$$\tilde{I}_\varepsilon(\hat{U}) \leqslant \Phi_\nu(\hat{U}) + o_\varepsilon(1)$$

where $\nu = \max_{x\in\overline{B_1}} V(x)$ and Φ_ν is defined in (8) . By shrinking B_1, if necessary, we can assume that $\Phi_\nu(\hat{U})$ is negative, so we obtain (14).

In order to prove (15), let us first observe that there exists $\sigma > 0$ such that $V(x) < 1 - \sigma$ for every $x \in \partial B_0^\varepsilon \cap E$. Then,

$$\begin{aligned}\tilde{I}_\varepsilon\big(\gamma_t(\cdot - \xi)\big) \leqslant{}& \int\limits_{B(0,1/\sqrt{\varepsilon})} \left(\frac{1}{2}\big|\nabla\gamma_t(x)\big|^2 + \frac{1}{2}V\big(\varepsilon(x+\xi)\big)\gamma_t^2(x) - F\big(\gamma_t(x)\big)\right) \\ &+ \int\limits_{B(0,1/\sqrt{\varepsilon})^c} \left(\frac{1}{2}\big|\nabla\gamma_t(x)\big|^2 + \frac{1}{2}\alpha_2\gamma_t^2(x)\right).\end{aligned}$$

Again by the exponential decay of γ_t, the second right term tends to zero as $\varepsilon \to 0$. Observe also that this convergence is uniform in t, since the exponential decay is uniform in t. By using dominated convergence theorem,

$$\tilde{I}_\varepsilon\big(\gamma_t(\cdot - \xi)\big) \leqslant \Phi_{1-\sigma}(\gamma_t) + o_\varepsilon(1).$$

Finally, since $\Phi_{1-\sigma}(u) < \Phi(u)$ for any $u \neq 0$, we have that

$$\max_{t\in[0,1]} \Phi_{1-\sigma}(\gamma_t) < \max_{t\in[0,1]} \Phi(\gamma_t) = m. \qquad \square$$

We now give a first estimate on the min–max values m_ε.

Proposition 3.2. *We have that*

$$\limsup_{\varepsilon\to 0} m_\varepsilon \leqslant m.$$

Proof. By definition,

$$m_\varepsilon \leqslant \max_{u\in\mathcal{C}_\varepsilon} \tilde{I}_\varepsilon(u).$$

So, let us estimate this last term. In the following we take a sequence $\varepsilon = \varepsilon_n \to 0$, but we drop the sub-index n for the sake of clarity. For any $\varepsilon > 0$ sufficiently small, there exist $t_\varepsilon \in [0, 1]$, $\xi_\varepsilon \in \overline{B_0^\varepsilon} \cap E$ such that

$$\max_{u\in\mathcal{C}_\varepsilon}\tilde I_\varepsilon(u)=\tilde I_\varepsilon\big(\gamma_{t_\varepsilon}(\cdot-\xi_\varepsilon)\big)$$
$$\leqslant\int\limits_{B(0,1/\sqrt{\varepsilon})}\left(\frac12\big|\nabla\gamma_{t_\varepsilon}(x)\big|^2+\frac12 V\big(\varepsilon(x+\xi_\varepsilon)\big)\gamma_{t_\varepsilon}^2(x)-F\big(\gamma_{t_\varepsilon}(x)\big)\right)$$
$$+\int\limits_{B(0,1/\sqrt{\varepsilon})^c}\left(\frac12\big|\nabla\gamma_{t_\varepsilon}(x)\big|^2+\frac12\alpha_2\gamma_{t_\varepsilon}^2(x)\right).$$

Up to a subsequence we can assume that $t_\varepsilon\to t_0\in[0,1]$ and $\varepsilon\xi_\varepsilon\to x_0\in\overline{B_0}\cap E$. Therefore, by the uniform exponential decay of γ_t and dominated convergence theorem, we get

$$\tilde I_\varepsilon\big(\gamma_{t_\varepsilon}(\cdot-\xi_\varepsilon)\big)\to\Phi_{V(x_0)}(\gamma_{t_0}).$$

Observe now that $V(x_0)\leqslant 1$ and then

$$\Phi_{V(x_0)}(\gamma_{t_0})\leqslant\max_{t\in[0,1]}\Phi(\gamma_t)=m.\qquad\square$$

At this point, our purpose is to give an analogous estimate from below on m_ε. In order to do that, we compare it with another min–max value, which we define now.

Let us define π_E as the orthogonal projection on E and we set $h_\varepsilon:\mathbb{R}^N\to E$ defined as $h_\varepsilon(x)=\pi_E(x)\chi_{B_3^\varepsilon}(x)$, where $\chi_{B_3^\varepsilon}$ is the characteristic function related to B_3^ε. Let us define a barycenter type map $\beta_\varepsilon:H^1(\mathbb{R}^N)\setminus\{0\}\to E$ such that for any $u\in H^1(\mathbb{R}^N)\setminus\{0\}$

$$\beta_\varepsilon(u)=\frac{\int_{\mathbb{R}^N}h_\varepsilon(x)u^2\,dx}{\int_{\mathbb{R}^N}u^2\,dx}.$$

For a fixed $\delta>0$ sufficiently small, we define

$$\varXi_\varepsilon=\left\{\Sigma\subset H^1\big(\mathbb{R}^N\big)\setminus\{0\}:\ \begin{array}{l}\Sigma\text{ is connected and compact}\\ \exists u_0,u_1\in\Sigma\text{ s.t. }\|u_0\|\leqslant\delta,\ \tilde I_\varepsilon(u_1)<0\\ \forall u\in\Sigma,\ \beta_\varepsilon(u)=0.\end{array}\right\}. \tag{16}$$

Let us observe that we need to require that $0\notin\Sigma$ because the barycenter β_ε is not well defined in 0. We also define the corresponding min–max value

$$b_\varepsilon=\inf_{\Sigma\in\varXi_\varepsilon}\max_{u\in\Sigma}\tilde I_\varepsilon(u). \tag{17}$$

Observe that, since $\tilde I_\varepsilon\geqslant\Phi_{\alpha_1}$, we have

$$b_\varepsilon\geqslant m_{\alpha_1}>0.$$

Lemma 3.3. *There exists $\varepsilon_0>0$ such that for $\varepsilon\in(0,\varepsilon_0)$,*

$$m_\varepsilon\geqslant b_\varepsilon.$$

Proof. Let us take $t_0 > 0$ sufficiently small and an arbitrary $\eta \in \Gamma_\varepsilon$. For any $t \in [t_0, 1]$, we define $\psi_t^\varepsilon : \overline{B_0^\varepsilon} \cap E \to E$ such that

$$\psi_t^\varepsilon(\xi) = \beta_\varepsilon\big(\eta\big(\gamma_t(\cdot - \xi)\big)\big).$$

Let us observe that, by the properties of $\eta \in \Gamma_\varepsilon$, $\eta(\gamma_t(\cdot - \xi)) \neq 0$, for all $t \in [t_0, 1]$ and $\xi \in \overline{B_0^\varepsilon} \cap E$, so ψ_t^ε is well defined. Moreover, $\|\gamma_{t_0}\|$ can be made arbitrary small by taking smaller t_0.

Moreover, by the exponential decay of γ_t,

$$\psi_t^\varepsilon(\xi) \to \xi \quad \text{uniformly in } \partial B_0^\varepsilon \cap E \text{ and } t \in [t_0, 1], \text{ as } \varepsilon \to 0.$$

Therefore we can choose ε_0 small enough (independent of η) so that if $\varepsilon \in (0, \varepsilon_0)$,

$$\deg\big(\psi_t^\varepsilon, B_0^\varepsilon \cap E, 0\big) = \deg\big(\mathrm{Id}, B_0^\varepsilon \cap E, 0\big) = 1 \quad \text{for all } t \in [t_0, 1].$$

We can conclude that for every $t \in [t_0, 1]$, there exists $\xi_t \in B_0^\varepsilon \cap E$ such that $\psi_t^\varepsilon(\xi_t) = 0$. Moreover there exists a connected and compact set $\Upsilon \subset [t_0, 1] \times (B_0^\varepsilon \cap E)$ that takes all values in $[t_0, 1]$ and such that $\psi_t^\varepsilon(\xi) = 0$ for all $(t, \xi) \in \Upsilon$ (see [24,27]).

Then, the set

$$\Sigma = \big\{\eta\big(\gamma_t(\cdot - \xi)\big) \colon (t, \xi) \in \Upsilon\big\}$$

belongs to Ξ_ε and $\Sigma \subset \eta(\mathcal{C}_\varepsilon)$. There follows

$$\max_{u \in \mathcal{C}_\varepsilon} \tilde{I}_\varepsilon\big(\eta(u)\big) \geqslant \max_{v \in \Sigma} \tilde{I}_\varepsilon(v) \geqslant b_\varepsilon,$$

which concludes the proof. □

The proof of the following proposition contains the main difficulties of the paper. For that reason, it is postponed to Section 4.

Proposition 3.4. *We have that*

$$\liminf_{\varepsilon \to 0} b_\varepsilon \geqslant m.$$

By putting together Proposition 3.2, Lemma 3.3 and Proposition 3.4, we obtain the following result.

Proposition 3.5. *We have that*

$$\lim_{\varepsilon \to 0} m_\varepsilon = m.$$

The following theorem is the main result of this section.

Theorem 3.6. *There exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ there exists a positive solution u_ε of the problem* (7). *Moreover, $\tilde{I}_\varepsilon(u_\varepsilon) = m_\varepsilon$.*

Proof. From Propositions 3.1 and 3.5 we get that for small values of ε, $m_\varepsilon > \max_{\partial\mathcal{C}_\varepsilon} \tilde{I}_\varepsilon$. Moreover, recall that $\tilde{I}_\varepsilon$ satisfies the (PS) condition (see Proposition 2.2). Therefore, classical min–max theory implies that m_ε is a critical value of $\tilde{I}_\varepsilon$; let us denote u_ε a critical point. Finally the fact that u_ε is positive follows from the maximum principle. □

4. Proof of Proposition 3.4

This section is devoted to prove that $\liminf_{\varepsilon\to 0} b_\varepsilon \geqslant m$ (recall the definitions (9), (10), (16), (17)). This proof will be divided in several lemmas and propositions. First, we show that the min–max value b_ε gives rise to a certain solution u_ε of a problem with a Lagrange multiplier $\lambda_\varepsilon \in E$. After that, some estimates are needed on u_ε and λ_ε. In particular, the asymptotic behavior of u_ε is studied by using concentration-compactness arguments. We finish the proof by discussing two possible cases which depend on the location of the mass of u_ε.

As a first step, we prove the existence of the solutions u_ε.

Lemma 4.1. *There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, there exist $u_\varepsilon \in H^1(\mathbb{R}^N)$, with $\beta_\varepsilon(u_\varepsilon) = 0$, and $\lambda_\varepsilon \in E$ such that*

$$-\Delta u_\varepsilon + V(\varepsilon x)u_\varepsilon = g_\varepsilon(x, u_\varepsilon) + \lambda_\varepsilon \cdot h_\varepsilon(x)u_\varepsilon \tag{18}$$

and

$$\tilde{I}_\varepsilon(u_\varepsilon) = b_\varepsilon.$$

Moreover, the sequence $\{u_\varepsilon\}$ is bounded in $H^1(\mathbb{R}^N)$.

Proof. Let $\varepsilon > 0$ be fixed. By classical min–max theory, there exists a sequence $\{u_n\} \subset H^1(\mathbb{R}^N)$ which is a constrained (PS) sequence at level b_ε, namely, there exists $\{\lambda_n\} \subset E$ such that

$$\tilde{I}_\varepsilon(u_n) \to b_\varepsilon, \quad \text{as } n \to +\infty, \tag{19}$$

$$\tilde{I}'_\varepsilon(u_n) - \frac{\lambda_n \cdot h_\varepsilon(x)u_n}{\int_{\mathbb{R}^N} u_n^2} \to 0, \quad \text{as } n \to +\infty. \tag{20}$$

Since $\beta_\varepsilon(u_n) = 0$, by (19) and (20), repeating the arguments of Proposition 2.2, we get that $\{u_n\}$ is bounded in the H^1-norm (uniformly with respect to ε) and, therefore, up to a subsequence, it converges weakly to some $u_\varepsilon \in H^1(\mathbb{R}^N)$. This convergence is actually strong arguing as in the proof of Proposition 2.2 and choosing R big enough such that $\phi_R h_\varepsilon = 0$. Since $u_\varepsilon \neq 0$, also the sequence λ_n is bounded, and this concludes the proof. □

The remaining of the proof of Proposition 3.4 is based on the study of the asymptotic behavior of the sequence of solutions u_ε.

Lemma 4.2. *There holds $u_\varepsilon \chi_{B_2^\varepsilon} \nrightarrow 0$ in $L^2(\mathbb{R}^N)$, as $\varepsilon \to 0$.*

Proof. Since u_ε is a solution of (18) with $\beta_\varepsilon(u_\varepsilon)=0$, multiplying (18) by u_ε, integrating and using (fg), for a fixed sufficiently small $\delta>0$, we have

$$\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 + V(\varepsilon x)u_\varepsilon^2 = \int_{\mathbb{R}^N} g_\varepsilon(x,u_\varepsilon)u_\varepsilon \leqslant \int_{\mathbb{R}^N} (a+\delta)u_\varepsilon^2 + C\int_{B_2^\varepsilon} u_\varepsilon^{p+1}.$$

Then

$$\|u_\varepsilon\|^2_{L^{p+1}(B_2^\varepsilon)} \leqslant C\|u_\varepsilon\|^2 \leqslant C\|u_\varepsilon\|^{p+1}_{L^{p+1}(B_2^\varepsilon)},$$

and

$$u_\varepsilon \chi_{B_2^\varepsilon} \nrightarrow 0, \quad \text{in } L^{p+1}(\mathbb{R}^N).$$

Now, by the boundedness of $\{u_\varepsilon\}$ in $H^1(\mathbb{R}^N)$ and so in $L^s(\mathbb{R}^N)$, for a certain $s>p+1$, we can conclude by interpolation: indeed, for a suitable $\alpha<1$

$$0<c\leqslant \|u_\varepsilon\|_{L^{p+1}(B_2^\varepsilon)} \leqslant \|u_\varepsilon\|^\alpha_{L^2(B_2^\varepsilon)}\|u_\varepsilon\|^{1-\alpha}_{L^s(B_2^\varepsilon)} \leqslant C\|u_\varepsilon\|^\alpha_{L^2(B_2^\varepsilon)}. \qquad \square$$

Lemma 4.3. *We have that* $\|u_\varepsilon\|_{H^1((B_4^\varepsilon)^c)} \to 0$ *as* $\varepsilon\to 0$.

Proof. Let $\phi_\varepsilon:\mathbb{R}^N\to\mathbb{R}$ be a smooth function such that

$$\phi_\varepsilon(x)=\begin{cases} 0 & \text{in } B_3^\varepsilon,\\ 1 & \text{in } (B_4^\varepsilon)^c,\end{cases}$$

with $0\leqslant\phi_\varepsilon\leqslant 1$ and $|\nabla\phi_\varepsilon|\leqslant C\varepsilon$.

By Lemma 4.1, since $\phi_\varepsilon h_\varepsilon=0$, we have that

$$\tilde I_\varepsilon'(u_\varepsilon)[\phi_\varepsilon u_\varepsilon]=0,$$

namely, by definition of g_ε,

$$\int_{\mathbb{R}^N} \big(|\nabla u_\varepsilon|^2 + V(\varepsilon x)u_\varepsilon^2\big)\phi_\varepsilon + \int_{\mathbb{R}^N} u_\varepsilon\nabla u_\varepsilon\cdot\nabla\phi_\varepsilon = \int_{\mathbb{R}^N} g_\varepsilon(x,u_\varepsilon)u_\varepsilon\phi_\varepsilon \leqslant \int_{\mathbb{R}^N} a u_\varepsilon^2\phi_\varepsilon,$$

and so we can conclude observing that

$$\int_{(B_4^\varepsilon)^c} |\nabla u_\varepsilon|^2 + u_\varepsilon^2 \leqslant C\varepsilon. \qquad \square$$

Lemma 4.4. *We have that* $\lambda_\varepsilon = O(\varepsilon)$.

Proof. In the sequel we can assume that $\lambda_\varepsilon \neq 0$, otherwise the lemma is proved. Let us denote $\tilde{\lambda}_\varepsilon = \lambda_\varepsilon / |\lambda_\varepsilon|$.

Let $\phi_\varepsilon : \mathbb{R}^N \to \mathbb{R}$ be a smooth function such that

$$\phi_\varepsilon(x) = \begin{cases} 1 & \text{in } B_2^\varepsilon, \\ 0 & \text{in } (B_3^\varepsilon)^c, \end{cases}$$

with $0 \leqslant \phi_\varepsilon \leqslant 1$ and $|\nabla \phi_\varepsilon| \leqslant C\varepsilon$.

We follow an idea of [17,18]. By regularity arguments $u_\varepsilon \in H^2(\mathbb{R}^N)$ and then we are allowed to multiply (18) by $\phi_\varepsilon \partial_{\tilde{\lambda}_\varepsilon} u_\varepsilon$ and to integrate by parts. Then

$$\begin{aligned} &\int_{B_3^\varepsilon} \big[\nabla u_\varepsilon \cdot \nabla(\partial_{\tilde{\lambda}_\varepsilon} u_\varepsilon)\big]\phi_\varepsilon + \int_{B_3^\varepsilon \setminus B_2^\varepsilon} (\nabla u_\varepsilon \cdot \nabla \phi_\varepsilon)(\partial_{\tilde{\lambda}_\varepsilon} u_\varepsilon) \\ &\qquad + \int_{B_3^\varepsilon} V(\varepsilon x) u_\varepsilon (\partial_{\tilde{\lambda}_\varepsilon} u_\varepsilon)\phi_\varepsilon - \int_{B_3^\varepsilon} g_\varepsilon(x, u_\varepsilon)(\partial_{\tilde{\lambda}_\varepsilon} u_\varepsilon)\phi_\varepsilon \\ &\quad = \int_{B_3^\varepsilon} \big(\lambda_\varepsilon \cdot h_\varepsilon(x)\big) u_\varepsilon (\partial_{\tilde{\lambda}_\varepsilon} u_\varepsilon)\phi_\varepsilon. \end{aligned} \tag{21}$$

Let us evaluate each term of the previous equality. We have

$$0 = \int_{\mathbb{R}^N} \partial_{\tilde{\lambda}_\varepsilon}\big[|\nabla u_\varepsilon|^2 \phi_\varepsilon\big] = 2\int_{\mathbb{R}^N} \big[\nabla u_\varepsilon \cdot \nabla(\partial_{\tilde{\lambda}_\varepsilon} u_\varepsilon)\big]\phi_\varepsilon + \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 \partial_{\tilde{\lambda}_\varepsilon} \phi_\varepsilon$$

and so

$$\int_{B_3^\varepsilon} \big[\nabla u_\varepsilon \cdot \nabla(\partial_{\tilde{\lambda}_\varepsilon} u_\varepsilon)\big]\phi_\varepsilon = -\frac{1}{2} \int_{B_3^\varepsilon \setminus B_2^\varepsilon} |\nabla u_\varepsilon|^2 \partial_{\tilde{\lambda}_\varepsilon} \phi_\varepsilon = O(\varepsilon). \tag{22}$$

Easily we have

$$\int_{B_3^\varepsilon \setminus B_2^\varepsilon} (\nabla u_\varepsilon \cdot \nabla \phi_\varepsilon)(\partial_{\tilde{\lambda}_\varepsilon} u_\varepsilon) = O(\varepsilon). \tag{23}$$

Analogously, we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} \partial_{\tilde{\lambda}_\varepsilon}\big[V(\varepsilon x) u_\varepsilon^2 \phi_\varepsilon\big] \\ &= \varepsilon \int_{\mathbb{R}^N} \big(\partial_{\tilde{\lambda}_\varepsilon} V(\varepsilon x)\big) u_\varepsilon^2 \phi_\varepsilon + 2\int_{\mathbb{R}^N} V(\varepsilon x)(\partial_{\tilde{\lambda}_\varepsilon} u_\varepsilon) u_\varepsilon \phi_\varepsilon + \int_{\mathbb{R}^N} V(\varepsilon x) u_\varepsilon^2 (\partial_{\tilde{\lambda}_\varepsilon} \phi_\varepsilon) \end{aligned}$$

and so

$$\int_{B_3^\varepsilon} V(\varepsilon x)u_\varepsilon(\partial_{\tilde{\lambda}_\varepsilon} u_\varepsilon)\phi_\varepsilon = -\frac{\varepsilon}{2}\int_{B_3^\varepsilon} \big(\partial_{\tilde{\lambda}_\varepsilon} V(\varepsilon x)\big)u_\varepsilon^2\phi_\varepsilon - \frac{1}{2}\int_{B_3^\varepsilon\setminus B_2^\varepsilon} V(\varepsilon x)u_\varepsilon^2(\partial_{\tilde{\lambda}_\varepsilon}\phi_\varepsilon) = O(\varepsilon). \tag{24}$$

Moreover, since by the definition of G_ε,

$$\partial_{\tilde{\lambda}_\varepsilon} G_\varepsilon(x,u_\varepsilon) = \varepsilon\partial_{\tilde{\lambda}_\varepsilon}\chi(\varepsilon x)\big(F(u_\varepsilon) - \tilde{F}(u_\varepsilon)\big) + g_\varepsilon(x,u_\varepsilon)\partial_{\tilde{\lambda}_\varepsilon} u_\varepsilon,$$

we have

$$\begin{aligned}
0 &= \int_{\mathbb{R}^N} \partial_{\tilde{\lambda}_\varepsilon}\big[G_\varepsilon(x,u_\varepsilon)\phi_\varepsilon\big] \\
&= \varepsilon\int_{\mathbb{R}^N} \big(F(u_\varepsilon) - \tilde{F}(u_\varepsilon)\big)\big(\partial_{\tilde{\lambda}_\varepsilon}\chi(\varepsilon x)\big)\phi_\varepsilon + \int_{\mathbb{R}^N} g_\varepsilon(x,u_\varepsilon)(\partial_{\tilde{\lambda}_\varepsilon} u_\varepsilon)\phi_\varepsilon + \int_{\mathbb{R}^N} G_\varepsilon(x,u_\varepsilon)(\partial_{\tilde{\lambda}_\varepsilon}\phi_\varepsilon)
\end{aligned}$$

so it follows

$$\begin{aligned}
\int_{B_3^\varepsilon} g_\varepsilon(x,u_\varepsilon)(\partial_{\tilde{\lambda}_\varepsilon} u_\varepsilon)\phi_\varepsilon &= -\varepsilon\int_{B_3^\varepsilon} \big(F(u_\varepsilon) - \tilde{F}(u_\varepsilon)\big)\big(\partial_{\tilde{\lambda}_\varepsilon}\chi(\varepsilon x)\big)\phi_\varepsilon \\
&\quad - \int_{B_3^\varepsilon\setminus B_2^\varepsilon} G_\varepsilon(x,u_\varepsilon)(\partial_{\tilde{\lambda}_\varepsilon}\phi_\varepsilon) = O(\varepsilon).
\end{aligned} \tag{25}$$

Finally

$$\begin{aligned}
0 &= \int_{B_3^\varepsilon} \partial_{\tilde{\lambda}_\varepsilon}\big[\big(\lambda_\varepsilon\cdot h_\varepsilon(x)\big)u_\varepsilon^2\phi_\varepsilon\big] \\
&= |\lambda_\varepsilon|\int_{B_3^\varepsilon} u_\varepsilon^2\phi_\varepsilon + 2\int_{B_3^\varepsilon} \big(\lambda_\varepsilon\cdot h_\varepsilon(x)\big)u_\varepsilon(\partial_{\tilde{\lambda}_\varepsilon} u_\varepsilon)\phi_\varepsilon + \int_{B_3^\varepsilon\setminus B_2^\varepsilon} \big(\lambda_\varepsilon\cdot h_\varepsilon(x)\big)u_\varepsilon^2(\partial_{\tilde{\lambda}_\varepsilon}\phi_\varepsilon)
\end{aligned}$$

and then

$$\int_{B_3^\varepsilon} \big(\lambda_\varepsilon\cdot h_\varepsilon(x)\big)u_\varepsilon(\partial_{\tilde{\lambda}_\varepsilon} u_\varepsilon)\phi_\varepsilon = -\frac{1}{2}|\lambda_\varepsilon|\int_{B_3^\varepsilon} u_\varepsilon^2\phi_\varepsilon + O(\varepsilon). \tag{26}$$

By (21)–(26) and by Lemma 4.2, we conclude. □

In what follows we consider a sequence $\varepsilon_k \to 0$, that we still denote by ε. Therefore, passing to a subsequence if necessary, we can assume that

$$\frac{\lambda_\varepsilon}{\varepsilon} \to \bar{\lambda} \in E.$$

Let us define

$$H_\varepsilon = \left\{ x \in \mathbb{R}^N \colon \bar{\lambda} \cdot x \leqslant \frac{\alpha_1}{2\varepsilon} \right\}.$$

The next proposition gives a complete description of $u_\varepsilon|_{H_\varepsilon}$ as $\varepsilon \to 0$.

Proposition 4.5. *Passing to a subsequence, if necessary, there exist $n \in \mathbb{N}$, $\bar{c} > 0$ and, for any $i = 1, \dots, n$, $y_\varepsilon^i \in B_2^\varepsilon \cap H_\varepsilon$, $\bar{y}_i \in B_2$ and $u_i \in H^1(\mathbb{R}^N) \setminus \{0\}$ positive solutions of*

$$-\Delta u_i + V(\bar{y}_i) u_i = g(\bar{y}_i, u_i) + \bar{\lambda} \cdot \bar{y}_i u_i,$$

such that

$$\begin{aligned}
&\varepsilon y_\varepsilon^i \to \bar{y}_i, \\
&\left| y_\varepsilon^i - y_\varepsilon^j \right| \to \infty, \quad \textit{if } i \neq j, \\
&u_\varepsilon\left(\cdot + y_\varepsilon^i\right) \rightharpoonup u_i, \quad \textit{weakly in } H^1\left(\mathbb{R}^N\right), \\
&\|u_i\| \geqslant \bar{c}, \\
&\left\| u_\varepsilon - \sum_{i=1}^n u_i\left(\cdot - y_\varepsilon^i\right) \right\|_{H^1(H_\varepsilon)} \to 0.
\end{aligned}$$

Let us observe that if $\bar{\lambda} = 0$, $H_\varepsilon = \mathbb{R}^N$ and Proposition 4.5 follows from Proposition 4.2 of [22]. In the general case, the effect of the Lagrange multiplier is not negligible, and some more technical work is needed. The proof is postponed to Subsection A.2 in Appendix A.

With that description of the asymptotic behavior of u_ε we are ready to prove that $\liminf_{\varepsilon \to 0} \tilde{I}_\varepsilon(u_\varepsilon) \geqslant m$. We distinguish two possible situations.

Case 1: $\bar{\lambda} \cdot \bar{y}_i \geqslant 0$, for all $i = 1, \dots, n$.
Since $\beta_\varepsilon(u_\varepsilon) = 0$, we have that

$$0 = \int\limits_{H_\varepsilon} \lambda_\varepsilon \cdot h_\varepsilon(x) u_\varepsilon^2 + \int\limits_{(H_\varepsilon)^c} \lambda_\varepsilon \cdot h_\varepsilon(x) u_\varepsilon^2.$$

By Proposition 4.5 and since $\bar{\lambda} \cdot \bar{y}_i \geqslant 0$, for all $i = 1, \dots, n$, we know that

$$\int\limits_{H_\varepsilon} \lambda_\varepsilon \cdot h_\varepsilon(x) u_\varepsilon^2 \to \sum_{i=1}^n \bar{\lambda} \cdot \bar{y}_i \int\limits_{\mathbb{R}^N} u_i^2 \geqslant 0,$$

whereas $\lambda_\varepsilon \cdot h_\varepsilon(x) \geqslant \frac{\alpha_1}{2\varepsilon}$ in $B_3^\varepsilon \setminus H_\varepsilon$. Therefore we have

$$\bar{\lambda} \cdot \bar{y}_i = 0, \quad \text{for all } i = 1, \dots, n,$$

and

$$\frac{\alpha_1}{2\varepsilon}\int_{B_3^\varepsilon\setminus H_\varepsilon} u_\varepsilon^2 \leqslant \int_{(H_\varepsilon)^c} \lambda_\varepsilon\cdot h_\varepsilon(x)u_\varepsilon^2 \to 0, \quad \text{as } \varepsilon\to 0. \tag{27}$$

With that information in hand, let us estimate the energy $\tilde{I}_\varepsilon(u_\varepsilon)$

$$\begin{aligned}\tilde{I}_\varepsilon(u_\varepsilon) = &\int_{B_2^\varepsilon\cap H_\varepsilon}\left[\frac{1}{2}\left(|\nabla u_\varepsilon|^2+V(\varepsilon x)u_\varepsilon^2\right)-G_\varepsilon(x,u_\varepsilon)\right]\\ &+\int_{B_2^\varepsilon\setminus H_\varepsilon}\left[\frac{1}{2}\left(|\nabla u_\varepsilon|^2+V(\varepsilon x)u_\varepsilon^2\right)-G_\varepsilon(x,u_\varepsilon)\right]\\ &+\int_{(B_2^\varepsilon)^c}\left[\frac{1}{2}\left(|\nabla u_\varepsilon|^2+V(\varepsilon x)u_\varepsilon^2\right)-G_\varepsilon(x,u_\varepsilon)\right].\end{aligned}$$

By Proposition 4.5, we have that

$$\int_{B_2^\varepsilon\cap H_\varepsilon}\left[\frac{1}{2}\left(|\nabla u_\varepsilon|^2+V(\varepsilon x)u_\varepsilon^2\right)-G_\varepsilon(x,u_\varepsilon)\right]=\sum_{i=1}^n J_{\bar{y}_i}(u_i)+o_\varepsilon(1).$$

Moreover, since $\{u_\varepsilon\}$ is a bounded sequence in $H^1(\mathbb{R}^N)$ and so also in $L^s(\mathbb{R}^N)$ (for a certain $s>p+1$), we can use interpolation and (27) to get

$$\int_{B_3^\varepsilon\setminus H_\varepsilon} u_\varepsilon^{p+1}\to 0, \quad \text{as } \varepsilon\to 0.$$

Then, we obtain

$$\int_{B_2^\varepsilon\setminus H_\varepsilon}\left[\frac{1}{2}\left(|\nabla u_\varepsilon|^2+V(\varepsilon x)u_\varepsilon^2\right)-G_\varepsilon(x,u_\varepsilon)\right]\geqslant o_\varepsilon(1).$$

Finally, by the definition of $G_\varepsilon(x,u)$, we have

$$\int_{(B_2^\varepsilon)^c}\left[\frac{1}{2}\left(|\nabla u_\varepsilon|^2+V(\varepsilon x)u_\varepsilon^2\right)-G_\varepsilon(x,u_\varepsilon)\right]\geqslant 0.$$

So, we get the estimate

$$\lim_{\varepsilon\to 0} b_\varepsilon=\lim_{\varepsilon\to 0}\tilde{I}_\varepsilon(u_\varepsilon)\geqslant\sum_{i=1}^n J_{\bar{y}_i}(u_i).$$

Since, by (g1), we have that $J_{\bar{y}_i}(u_i) \geqslant \Phi_{V(\bar{y}_i)}(u_i)$, for all $i = 1, \dots, n$, to conclude the proof, we have only to show that

$$\sum_{i=1}^{n} \Phi_{V(\bar{y}_i)}(u_i) \geqslant m.$$

This is trivially true if $n \geqslant 2$ by Lemma 2.4, since $\bar{y}_i \in B_2$. If, otherwise, $n = 1$, since $\beta_\varepsilon(u_\varepsilon) = 0$,

$$0 = \varepsilon \int_{B_3^\varepsilon} \pi_E(x) u_\varepsilon^2(x) = \varepsilon \int_{B_3^\varepsilon \cap H_\varepsilon} \pi_E(x) u_\varepsilon^2(x) + \varepsilon \int_{B_3^\varepsilon \setminus H_\varepsilon} \pi_E(x) u_\varepsilon^2(x).$$

By (27), the second right term of the last expression tends to 0. Moreover

$$\begin{aligned} \varepsilon \int_{B_3^\varepsilon \cap H_\varepsilon} \pi_E(x) u_\varepsilon^2(x) &= \varepsilon \int_{(B_3^\varepsilon \cap H_\varepsilon) - y_\varepsilon^1} \pi_E\big(x + y_\varepsilon^1\big) u_\varepsilon^2\big(x + y_\varepsilon^1\big) \\ &= \int_{(B_3^\varepsilon \cap H_\varepsilon) - y_\varepsilon^1} \pi_E\big(\varepsilon x + \varepsilon y_\varepsilon^1\big) u_\varepsilon^2\big(x + y_\varepsilon^1\big) \to \pi_E(\bar{y}_1) \int_{\mathbb{R}^N} u_1^2. \end{aligned}$$

Then $\bar{y}_1 \in E^\perp$, and we conclude

$$J_{\bar{y}_1}(u_1) \geqslant m_{V(\bar{y}_1)} \geqslant m.$$

Case 2: there exists at least an $i = 1, \dots, n$ such that $\bar{\lambda} \cdot \bar{y}_i < 0$.

Without lost of generality, we can assume that $\bar{\lambda} \cdot \bar{y}_1 < 0$. Let $s > 0$ such that $B(\bar{y}_1, 3s) \subset B_3$, with $\bar{y}_i \notin B(\bar{y}_1, 3s)$ for all $\bar{y}_i \neq \bar{y}_1$, and such that $\bar{\lambda} \cdot x < 0$, for all $x \in B(\bar{y}_1, 3s)$. We define $B_s^\varepsilon = \varepsilon^{-1} B(\bar{y}_1, s)$ and $B_{2s}^\varepsilon = \varepsilon^{-1} B(\bar{y}_1, 2s)$. By Proposition 4.5, there exists $c > 0$ such that

$$\int_{B_s^\varepsilon} u_\varepsilon^2 \geqslant c > 0. \tag{28}$$

Let ϕ_ε be a smooth function such that

$$\phi_\varepsilon(x) = \begin{cases} 1 & \text{if } x \in B_s^\varepsilon, \\ 0 & \text{if } x \in (B_{2s}^\varepsilon)^c, \end{cases}$$

with $0 \leqslant \phi_\varepsilon \leqslant 1$ and $|\nabla \phi_\varepsilon| \leqslant C\varepsilon$. Repeating the arguments of the proof of Lemma 4.4, we multiply (18) by $(\partial_{\tilde{\lambda}_\varepsilon} u_\varepsilon)\phi_\varepsilon$, where $\tilde{\lambda}_\varepsilon = \lambda_\varepsilon / |\lambda_\varepsilon|$. We have

$$\int_{B_{2s}^\varepsilon} \big[\nabla u_\varepsilon \cdot \nabla(\partial_{\tilde{\lambda}_\varepsilon} u_\varepsilon)\big]\phi_\varepsilon + \int_{B_{2s}^\varepsilon \setminus B_s^\varepsilon} (\nabla u_\varepsilon \cdot \nabla \phi_\varepsilon) \partial_{\tilde{\lambda}_\varepsilon} u_\varepsilon$$

$$+\int_{B_{2s}^{\varepsilon}} V(\varepsilon x)u_{\varepsilon}(\partial_{\tilde{\lambda}_{\varepsilon}}u_{\varepsilon})\phi_{\varepsilon} - \int_{B_{2s}^{\varepsilon}} g_{\varepsilon}(x,u_{\varepsilon})(\partial_{\tilde{\lambda}_{\varepsilon}}u_{\varepsilon})\phi_{\varepsilon}$$

$$= \int_{B_{2s}^{\varepsilon}} \big(\lambda_{\varepsilon}\cdot h_{\varepsilon}(x)\big)u_{\varepsilon}(\partial_{\tilde{\lambda}_{\varepsilon}}u_{\varepsilon})\phi_{\varepsilon}. \tag{29}$$

Let us evaluate each term of the previous equality. Since

$$\|u_{\varepsilon}\|_{H^1(B_{2s}^{\varepsilon}\setminus B_s^{\varepsilon})} \to 0,$$

we have

$$\int_{B_{2s}^{\varepsilon}} \big[\nabla u_{\varepsilon}\cdot\nabla(\partial_{\tilde{\lambda}_{\varepsilon}}u_{\varepsilon})\big]\phi_{\varepsilon} = -\frac{1}{2}\int_{B_{2s}^{\varepsilon}\setminus B_s^{\varepsilon}} |\nabla u_{\varepsilon}|^2\partial_{\tilde{\lambda}_{\varepsilon}}\phi_{\varepsilon} = o(\varepsilon), \tag{30}$$

$$\int_{B_{2s}^{\varepsilon}\setminus B_s^{\varepsilon}} (\nabla u_{\varepsilon}\cdot\nabla\phi_{\varepsilon})\partial_{\tilde{\lambda}_{\varepsilon}}u_{\varepsilon} = o(\varepsilon). \tag{31}$$

Analogously, we have

$$\int_{B_{2s}^{\varepsilon}} V(\varepsilon x)u_{\varepsilon}(\partial_{\tilde{\lambda}_{\varepsilon}}u_{\varepsilon})\phi_{\varepsilon} = -\frac{\varepsilon}{2}\int_{B_{2s}^{\varepsilon}} \big(\partial_{\tilde{\lambda}_{\varepsilon}}V(\varepsilon x)\big)u_{\varepsilon}^2\phi_{\varepsilon} - \frac{1}{2}\int_{B_{2s}^{\varepsilon}\setminus B_s^{\varepsilon}} V(\varepsilon x)u_{\varepsilon}^2\partial_{\tilde{\lambda}_{\varepsilon}}\phi_{\varepsilon}$$

$$= -\frac{\varepsilon}{2}\int_{B_{2s}^{\varepsilon}} \big(\partial_{\tilde{\lambda}_{\varepsilon}}V(\varepsilon x)\big)u_{\varepsilon}^2\phi_{\varepsilon} + o(\varepsilon). \tag{32}$$

Observe that $\partial_{\tilde{\lambda}_{\varepsilon}}\chi(\varepsilon x) \geqslant 0$ for all $x \in B_{2s}^{\varepsilon}$; this is the key point of our estimates in this case. Then, by $(\tilde{\mathrm{f}}1)$ we get that

$$\int_{B_{2s}^{\varepsilon}} g_{\varepsilon}(x,u_{\varepsilon})(\partial_{\tilde{\lambda}_{\varepsilon}}u_{\varepsilon})\phi_{\varepsilon} = -\varepsilon\int_{B_{2s}^{\varepsilon}} \big(F(u_{\varepsilon})-\tilde{F}(u_{\varepsilon})\big)\big(\partial_{\tilde{\lambda}_{\varepsilon}}\chi(\varepsilon x)\big)\phi_{\varepsilon} - \int_{B_{2s}^{\varepsilon}\setminus B_s^{\varepsilon}} G_{\varepsilon}(x,u_{\varepsilon})\partial_{\tilde{\lambda}_{\varepsilon}}\phi_{\varepsilon}$$

$$\leqslant o(\varepsilon). \tag{33}$$

Finally

$$\int_{B_{2s}^{\varepsilon}} \big(\lambda_{\varepsilon}\cdot h_{\varepsilon}(x)\big)u_{\varepsilon}(\partial_{\tilde{\lambda}_{\varepsilon}}u_{\varepsilon})\phi_{\varepsilon} = -\frac{1}{2}|\lambda_{\varepsilon}|\int_{B_{2s}^{\varepsilon}} u_{\varepsilon}^2\phi_{\varepsilon} + o(\varepsilon). \tag{34}$$

Therefore, by (28)–(34), we obtain the inequality

$$c\big(|\bar{\lambda}|+o_\varepsilon(1)\big) \leqslant \frac{|\lambda_\varepsilon|}{\varepsilon}\int\limits_{B_{2s}^\varepsilon} u_\varepsilon^2\phi_\varepsilon \leqslant \int\limits_{B_{2s}^\varepsilon}\big(\partial_{\tilde{\lambda}_\varepsilon}V(\varepsilon x)\big)u_\varepsilon^2\phi_\varepsilon + o_\varepsilon(1)$$
$$\leqslant C\max_{x\in B_3}\big|\nabla V(x)\big| + o_\varepsilon(1).$$

We can choose B_3 sufficiently small such that, for a suitable $\bar{\delta}>0$, we have that $|\bar{\lambda}|<\bar{\delta}$ and

$$B_3^\varepsilon \subset H_\varepsilon.$$

Now we can estimate $\tilde{I}_\varepsilon(u_\varepsilon)$. We have

$$\tilde{I}_\varepsilon(u_\varepsilon) = \int\limits_{B_2^\varepsilon}\left[\frac{1}{2}\big(|\nabla u_\varepsilon|^2 + V(\varepsilon x)u_\varepsilon^2\big) - G_\varepsilon(x,u_\varepsilon)\right]$$
$$+ \int\limits_{(B_2^\varepsilon)^c}\left[\frac{1}{2}\big(|\nabla u_\varepsilon|^2 + V(\varepsilon x)u_\varepsilon^2\big) - G_\varepsilon(x,u_\varepsilon)\right].$$

Since $B_3^\varepsilon \subset H_\varepsilon$, we can apply Proposition 4.5 to obtain

$$\int\limits_{B_2^\varepsilon}\left[\frac{1}{2}\big(|\nabla u_\varepsilon|^2 + V(\varepsilon x)u_\varepsilon^2\big) - G_\varepsilon(x,u_\varepsilon)\right] = \sum_{i=1}^n J_{\bar{y}_i}(u_i) + o_\varepsilon(1).$$

Moreover, by the definition of $G_\varepsilon(x,u)$, we have

$$\int\limits_{(B_2^\varepsilon)^c}\left[\frac{1}{2}\big(|\nabla u_\varepsilon|^2 + V(\varepsilon x)u_\varepsilon^2\big) - G_\varepsilon(x,u_\varepsilon)\right] \geqslant 0.$$

Then, we conclude that

$$\tilde{I}_\varepsilon(u_\varepsilon) \geqslant \sum_{i=1}^n J_{\bar{y}_i}(u_i) + o_\varepsilon(1).$$

As in Case 1, we conclude easily if $n \geqslant 1$. Assume now that $n=1$; since $B_3^\varepsilon \subset H_\varepsilon$, we can argue as in Case 1 to obtain

$$0 = \int\limits_{B_3^\varepsilon}\pi_E(x)u_\varepsilon^2(x) \to \pi_E(\bar{y}_1)\int\limits_{\mathbb{R}^N} u_1^2.$$

But this is in contradiction with the hypothesis of Case 2, namely, $\bar{\lambda}\cdot\bar{y}_1 < 0$.

5. Asymptotic behavior

In this section we will study the asymptotic behavior of the solution obtained in Section 3. As a consequence, u_ε will be actually a solution of (3): in this way we conclude the proof of Theorem 1.1.

Let us define u_ε the critical point of $\tilde{I}_\varepsilon$ at level m_ε, that is,

$$-\Delta u_\varepsilon + V(\varepsilon x)u_\varepsilon = g_\varepsilon(x, u_\varepsilon). \tag{35}$$

Moreover, Proposition 3.5 implies that $\tilde{I}_\varepsilon(u_\varepsilon) \to m$, as $\varepsilon \to 0$.

The following result gives a description of the behavior of u_ε, as $\varepsilon \to 0$.

Proposition 5.1. *Given a sequence $\varepsilon_j \to 0$, there exists a subsequence* (*still denoted by ε_j*) *and a sequence of points $y_{\varepsilon_j} \in \mathbb{R}^N$ such that*

$$\varepsilon_j y_{\varepsilon_j} \to 0,$$
$$\left\| u_{\varepsilon_j} - U(\cdot - y_{\varepsilon_j}) \right\| \to 0,$$

where $U \in \mathcal{S}$ (*see* (11)).

Proof. For the sake of clarity, let us write $\varepsilon = \varepsilon_j$. Our first tool is Proposition 4.2 of [22]; there exist $l \in \mathbb{N}$, sequences $\{y_\varepsilon^k\} \subset \mathbb{R}^N$, $\bar{y}_k \in B_2$, $U_k \in H^1(\mathbb{R}^N) \setminus \{0\}$ ($k = 1, \dots, l$) such that

$$\begin{aligned}
&\left|y_\varepsilon^k - y_\varepsilon^{k'}\right| \to +\infty, \quad \text{if } k \neq k', \\
&\varepsilon y_\varepsilon^k \to \bar{y}_k, \\
&\left\| u_\varepsilon - \sum_{k=1}^{l} U_k\left(\cdot - y_\varepsilon^k\right) \right\| \to 0, \\
&J'_{\bar{y}_k}(U_k) = 0, \\
&\tilde{I}_\varepsilon(u_\varepsilon) \to \sum_{k=1}^{l} J_{\bar{y}_k}(U_k).
\end{aligned}$$

For the definition of $J_{\bar{y}_k}$ see (12). Observe that $J_{\bar{y}_k}(U_k) \geqslant m_{V(\bar{y}_k)}$ since $J_{\bar{y}_k} \geqslant \Phi_{V(\bar{y}_k)}$. Moreover, Lemma 2.4 implies that $m_{V(\bar{y}_k)} \geqslant m - \delta$ for any $\bar{y}_k \in B_2$, where $\delta > 0$ can be taken arbitrary small by appropriately shrinking B_2: this implies that $l = 1$. So, the only thing that remains to be proved is that $\bar{y}_1 = 0$.

Our argument here has been used already in the previous sections, so we will be sketchy. By regularity arguments, $\{u_\varepsilon\} \subset H^2(\mathbb{R}^N)$ and is bounded. Choose $r > 0$ and ϕ_ε a cut-off function so that $\phi_\varepsilon(x) = 1$ in $B(y_\varepsilon^1, r\varepsilon^{-1})$ and $\phi_\varepsilon(x) = 0$ if $x \in B(y_\varepsilon^1, 2r\varepsilon^{-1})^c$, with $|\nabla\phi_\varepsilon| \leqslant C\varepsilon$. By multiplying (35) by $\phi_\varepsilon(x)\partial_\nu u_\varepsilon$ and integrating, we obtain

$$\frac{1}{2}\varepsilon \int\limits_{B(y_\varepsilon^1, \varepsilon^{-1}r)} \partial_\nu V(\varepsilon x) u_\varepsilon^2(x) - \varepsilon \int\limits_{B(y_\varepsilon^1, \varepsilon^{-1}r)} \partial_\nu \chi(\varepsilon x)\left[F\left(u_\varepsilon(x)\right) - \tilde{F}\left(u_\varepsilon(x)\right)\right] = o(\varepsilon). \tag{36}$$

If χ is $C^1(B(\bar{y}_1, r))$, we divide by ε and pass to the limit to obtain

$$\frac{1}{2}\partial_\nu V(\bar{y}_1)\int_{\mathbb{R}^N} U_1^2(x) - \partial_\nu \chi(\bar{y}_1)\int_{\mathbb{R}^N}\bigl[F\bigl(U_1(x)\bigr) - \tilde{F}\bigl(U_1(x)\bigr)\bigr] = 0. \tag{37}$$

We consider three different cases.

Case 1: $\bar{y}_1 \in B_1$.

Take $r > 0$ so that $B(\bar{y}_1, 2r) \subset B_1$. By (37), we get that $\partial_\nu V(\bar{y}_1) = 0$. Since ν is arbitrary, $\bar{y}_1$ is a critical point of V in B_1, and therefore $\bar{y}_1 = 0$.

Case 2: $\bar{y}_1 \in B_2 \setminus \overline{B_1}$.

In this case we will arrive to a contradiction. Take $r > 0$ so that $B(\bar{y}_1, 2r) \subset B_2 \setminus B_1$ and $\nu = \frac{1}{|\bar{y}_1|}\bar{y}_1$. By the definition of χ (see (6)), $\partial_\nu \chi(\bar{y}_1) = -1/(R_2 - R_1)$.

We now use the Pohozaev identity for U_1 to get

$$\begin{aligned}\int_{\mathbb{R}^N}\left(\frac{N-2}{2}|\nabla U_1|^2 + \frac{N}{2}V(\bar{y}_1)U_1^2\right) &= N\int_{\mathbb{R}^N}\chi(\bar{y}_1)F(U_1) + \bigl(1-\chi(\bar{y}_1)\bigr)\tilde{F}(U_1)\\ &\leqslant a\frac{N}{2}\int_{\mathbb{R}^N} U_1^2(x) + N\chi(\bar{y}_1)\int_{\mathbb{R}^N}\bigl[F\bigl(U_1(x)\bigr) - \tilde{F}\bigl(U_1(x)\bigr)\bigr]\end{aligned}$$

and so

$$c\int_{\mathbb{R}^N} U_1^2 \leqslant \int_{\mathbb{R}^N}\bigl[F\bigl(U_1(x)\bigr) - \tilde{F}\bigl(U_1(x)\bigr)\bigr].$$

So, it suffices to take $R_2 - R_1$ smaller, if necessary, to get a contradiction with (37).

Case 3: $\bar{y}_1 \in \partial B_1$.

Also in this case we obtain a contradiction. Indeed, observe that here $\chi(\bar{y}_1) = 1$, and so U_1 is a solution of

$$-\Delta U_1 + V(\bar{y}_1)U_1 = f(U_1).$$

Since $J_{\bar{y}_1}(U_1) = \Phi_{V(\bar{y}_1)}(U_1) = m$, Lemma 2.4 implies that $V(\bar{y}_1) = 1$. Then, by (5), there exists $\tau \in \mathbb{R}^N$ tangent to ∂B_1 at $\bar{y}_1$ such that $\partial_\tau V(\bar{y}_1) \neq 0$.

We now argue as above, with the exception that here χ is not C^1. However, it is a Lipschitz map so that (36) holds: let us choose $r < R_2 - R_1$ and $\nu = \tau$. Now we can write

$$\begin{aligned}&\left|\int_{B(y_\varepsilon^1, r/\varepsilon)}\partial_\tau\chi(\varepsilon x)\bigl[F\bigl(u_\varepsilon(x)\bigr) - \tilde{F}\bigl(u_\varepsilon(x)\bigr)\bigr]\right|\\ &\quad\leqslant \frac{1}{R_2 - R_1}\int_{B(0, r/\sqrt{\varepsilon})}\left[\frac{|x\cdot\tau|}{|x + y_\varepsilon^1|} + \frac{|y_\varepsilon^1\cdot\tau|}{|x + y_\varepsilon^1|}\right]\bigl[F\bigl(u_\varepsilon\bigl(x + y_\varepsilon^1\bigr)\bigr) - \tilde{F}\bigl(u_\varepsilon\bigl(x + y_\varepsilon^1\bigr)\bigr)\bigr]\end{aligned}$$

$$+\frac{1}{R_2-R_1}\int\limits_{r/\sqrt{\varepsilon}\leqslant|x|\leqslant r/\varepsilon}\frac{|(x+y_\varepsilon^1)\cdot\tau|}{|x+y_\varepsilon^1|}\left[F\left(u_\varepsilon\left(x+y_\varepsilon^1\right)\right)-\tilde{F}\left(u_\varepsilon\left(x+y_\varepsilon^1\right)\right)\right]\to 0.$$

In the above limit we have used again the dominated convergence theorem and the strong convergence of $u_\varepsilon(\cdot+y_\varepsilon^1)$. Then, we can divide by ε and pass to the limit in (36) to get

$$\frac{1}{2}\partial_\tau V(\bar{y}_1)\int\limits_{\mathbb{R}^N}U_1^2(x)=0,$$

a contradiction. □

Proof of Theorem 1.1. It suffices to show that u_ε is a solution of (1). Let us show that indeed $u_\varepsilon(x)\to 0$, as $\varepsilon\to 0$, uniformly in $x\in(B_1^\varepsilon)^c$. By Proposition 5.1 we obtain

$$\|u_\varepsilon\|_{H^1((B_0^\varepsilon)^c)}\leqslant\left\|u_\varepsilon-U(\cdot-y_\varepsilon)\right\|+\left\|U(\cdot-y_\varepsilon)\right\|_{H^1((B_0^\varepsilon)^c)}\to 0,$$

as $\varepsilon\to 0$. By using local L^∞ regularity of u_ε, given by standard bootstrap arguments, we obtain that for any $x\in(B_1^\varepsilon)^c$,

$$\|u_\varepsilon\|_{L^\infty(B(x,1))}\leqslant C\|u_\varepsilon\|_{H^1(B(x,2))}\leqslant C\|u_\varepsilon\|_{H^1((B_0^\varepsilon)^c)}\to 0.$$

This concludes the proof. □

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Appendix A

A.1. *Justification of (5) under (V3)*

Here we first show that under condition (V3), we can choose B_1 arbitrarily small so that (5) holds. This will be a consequence of the next proposition.

Proposition A.1. *Let $V: B(0,R)\subset\mathbb{R}^N\to\mathbb{R}$ be a C^{N-1} function with a unique critical point at* 0 *and assume that* $V(0)=1$. *Then, the following assertion is satisfied for almost every* $R\in(0,R_0)$:

$$\forall x\in\partial B(0,R)\ \textit{with}\ V(x)=1,\quad \partial_\tau V(x)\neq 0,\quad \textit{where}\ \tau\ \textit{is tangent to}\ \partial B(0,R)\ \textit{at}\ x. \tag{38}$$

Proof. The proof is an easy application of the Sard Lemma. Given $\delta \in (0, R_0)$, let us define the annulus $A = A(0; \delta, R_0)$. Let us consider the set

$$M = \{x \in A:\ V(x) = 1\}.$$

If M is empty, we are done. Otherwise, since V has no critical points in A, the implicit function theorem implies that M is an $N-1$ dimensional manifold with C^{N-1} regularity and a finite number of connected components; then, we can decompose $M = \bigcup_{i=1}^{n} M_i$, where M_i are connected.

Let us define the maps

$$\psi_i : M_i \to \mathbb{R}, \quad \psi(x) = |x|.$$

Since M_i is a C^{N-1} manifold, we can apply Sard Lemma: if we denote by $S_i \subset (\delta, R_0)$ the set of critical values of ψ_i, then S_i has 0 Lebesgue measure in $\mathbb{R}$. Define $S = \bigcup_{i=1}^{n} S_i$. It can be checked that for any $R \in (\delta, R_0) \setminus S$, (38) holds.

Now, it suffices to take $\delta_n \to 0$ and S^n the corresponding set of critical values. Clearly, $\bigcup_{n\in\mathbb{N}} S^n$ has also 0 Lebesgue measure and this finishes the proof. □

A.2. *Proof of Proposition 4.5*

The proof will be made in different steps.

Step 1. $u_\varepsilon \chi_{H_\varepsilon} \nrightarrow 0$ in the L^2-norm and in the L^{p+1}-norm.
Let $H'_\varepsilon = \{x \in \mathbb{R}^N : \bar{\lambda} \cdot x \leqslant \frac{\alpha_1}{3\varepsilon}\} \subset H_\varepsilon$. We will prove that

$$\int_{H'_\varepsilon} u_\varepsilon^2 \nrightarrow 0, \quad \text{as } \varepsilon \to 0. \tag{39}$$

Suppose by contradiction that

$$\int_{H'_\varepsilon} u_\varepsilon^2 \to 0, \quad \text{as } \varepsilon \to 0. \tag{40}$$

Since $\beta_\varepsilon(u_\varepsilon) = 0$ and $\bar{\lambda} \in E$, we have

$$0 = \int_{\mathbb{R}^N} \bar{\lambda} \cdot h_\varepsilon(x) u_\varepsilon^2 = \int_{(H'_\varepsilon)^c \cap B_3^\varepsilon} \bar{\lambda} \cdot x u_\varepsilon^2 + \int_{H'_\varepsilon \cap B_3^\varepsilon} \bar{\lambda} \cdot x u_\varepsilon^2 \geqslant \frac{\alpha_1}{3\varepsilon} \int_{(H'_\varepsilon)^c \cap B_3^\varepsilon} u_\varepsilon^2 + \int_{H'_\varepsilon \cap B_3^\varepsilon} \bar{\lambda} \cdot x u_\varepsilon^2.$$

Therefore

$$\frac{\alpha_1}{3\varepsilon} \int_{(H'_\varepsilon)^c \cap B_3^\varepsilon} u_\varepsilon^2 \leqslant \left| \int_{H'_\varepsilon \cap B_3^\varepsilon} \bar{\lambda} \cdot x u_\varepsilon^2 \right| \leqslant \frac{|\bar{\lambda}| R_3}{\varepsilon} \int_{H'_\varepsilon \cap B_3^\varepsilon} u_\varepsilon^2$$

and so

$$\int_{(H'_\varepsilon)^c\cap B_3^\varepsilon} u_\varepsilon^2 \to 0, \quad \text{as } \varepsilon \to 0.$$

This last formula, together with (40), implies that $u_\varepsilon \chi_{B_3^\varepsilon} \to 0$ in $L^2(\mathbb{R}^N)$ which is a contradiction with Lemma 4.2. This proves the first part of the claim of Step 1.

Let us now consider the L^{p+1} convergence. Take $\phi_\varepsilon : \mathbb{R}^N \to \mathbb{R}$ a smooth function such that

$$\phi_\varepsilon(x) = \begin{cases} 1 & \text{in } H'_\varepsilon, \\ 0 & \text{in } (H_\varepsilon)^c, \end{cases}$$

with $0 \leqslant \phi_\varepsilon \leqslant 1$ and $|\nabla\phi_\varepsilon| \leqslant C\varepsilon$. Multiplying (18) by $u_\varepsilon\phi_\varepsilon$ and integrating, we have

$$\int_{H_\varepsilon} |\nabla u_\varepsilon|^2\phi_\varepsilon + \int_{H_\varepsilon\setminus H'_\varepsilon} \nabla u_\varepsilon \cdot \nabla\phi_\varepsilon u_\varepsilon + \int_{H_\varepsilon} V(\varepsilon x)u_\varepsilon^2\phi_\varepsilon - \int_{H_\varepsilon} g_\varepsilon(x, u_\varepsilon)u_\varepsilon\phi_\varepsilon = \int_{H_\varepsilon} \lambda_\varepsilon \cdot h_\varepsilon(x)u_\varepsilon^2\phi_\varepsilon.$$

Therefore, by (fg), if $\delta > 0$ is sufficiently small, there exists $C_\delta > 0$, such that

$$\int_{H'_\varepsilon} |\nabla u_\varepsilon|^2 + \int_{H'_\varepsilon} \left(V(\varepsilon x) - \frac{\alpha_1}{2} - \delta\right)u_\varepsilon^2 \leqslant O(\varepsilon) + C_\delta \int_{H_\varepsilon} u_\varepsilon^{p+1},$$

and so the conclusion follows by (39).

Step 2. Passing to the limit by concentration-compactness.

We define $\tilde{u}_\varepsilon$ the even reflection of $u_\varepsilon|_{H_\varepsilon}$ with respect to ∂H_ε. Observe that $\{\tilde{u}_\varepsilon\}$ is bounded in $H^1(\mathbb{R}^N)$ and does not converge to 0 in $L^{p+1}(\mathbb{R}^N)$ by Step 1. Then, by concentration-compactness arguments (see [26, Lemma 1.1]), there exists $y_\varepsilon^1 \in \mathbb{R}^N$ such that

$$\int_{B(y_\varepsilon^1,1)} \tilde{u}_\varepsilon^2 \geqslant c > 0.$$

By the even symmetry of $\tilde{u}_\varepsilon$ and by Lemma 4.3, we can assume that $y_\varepsilon^1 \in H_\varepsilon \cap B_4^\varepsilon$. Therefore there exists $u_1 \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that $v_\varepsilon^1 = u_\varepsilon(\cdot + y_\varepsilon^1) \rightharpoonup u_1$, weakly in $H^1(\mathbb{R}^N)$.

Observe that v_ε^1 solves the equation

$$-\Delta v_\varepsilon^1 + V\big(\varepsilon x + \varepsilon y_\varepsilon^1\big)v_\varepsilon^1 = g\big(\varepsilon x + \varepsilon y_\varepsilon^1, v_\varepsilon^1\big) + \lambda_\varepsilon \cdot h_\varepsilon\big(x + y_\varepsilon^1\big)v_\varepsilon^1,$$

and so, passing to the limit, u_1 is a weak solution of

$$-\Delta u_1 + V(\bar{y}_1)u_1 = g(\bar{y}_1, u_1) + \bar{\lambda} \cdot \bar{y}_1 u_1,$$

where $\bar{y}_1 = \lim_{\varepsilon\to 0} \varepsilon y_\varepsilon^1$.

Since $y_\varepsilon^1 \in H_\varepsilon$, we have that $\bar\lambda \cdot \bar y_1 \leqslant \alpha_1/2$ and so $\bar y_1 \in B_2$ (otherwise u_1 should be 0) and, by (fg), we easily get that there exists $c > 0$ such that $c \leqslant \|u_1\|$. Moreover, observe that

$$\|u_\varepsilon\| \geqslant \|u_1\|.$$

Let us define $w_\varepsilon^1 = u_\varepsilon - u_1(\cdot - y_\varepsilon^1)$. We consider two possibilities: either $\|w_\varepsilon^1\|_{H^1(H_\varepsilon)} \to 0$ or not. In the first case the proposition should be proved taking $n = 1$. In the second case, there are still two sub-cases: either $\|w_\varepsilon^1\|_{L^{p+1}(H_\varepsilon)} \to 0$ or not.

Step 3. Assume that $\|w_\varepsilon^1\|_{L^{p+1}(H_\varepsilon)} \nrightarrow 0$.

In such case, we can repeat the previous argument to the sequence $\{w_\varepsilon^1\}$: we take $\tilde w_\varepsilon^1$ its even reflection with respect to ∂H_ε and apply [26, Lemma 1.1]; there exists $y_\varepsilon^2 \in H_\varepsilon$ such that

$$\int\limits_{B(y_\varepsilon^2,1)} \left(\tilde w_\varepsilon^1\right)^2 \geqslant c > 0.$$

Therefore, as above, there exists $u_2 \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that $v_\varepsilon^2 = w_\varepsilon^1(\cdot + y_\varepsilon^2) \rightharpoonup u_2$, weakly in $H^1(\mathbb{R}^N)$. Moreover, $|y_\varepsilon^1 - y_\varepsilon^2| \to +\infty$, $\varepsilon y_\varepsilon^2 \to \bar y_2 \in B_2$ and

$$-\Delta u_2 + V(\bar y_2)u_2 = g(\bar y_2, u_2) + \bar\lambda \cdot \bar y_2 u_2,$$

and $\|u_2\| \geqslant c > 0$. Moreover, by weak convergence,

$$\|u_\varepsilon\|^2 \geqslant \|u_1\|^2 + \|u_2\|^2.$$

Let us define $w_\varepsilon^2 = w_\varepsilon^1 - u_2(\cdot - y_\varepsilon^2) = u_\varepsilon - u_1(\cdot - y_\varepsilon^1) - u_2(\cdot - y_\varepsilon^2)$. Again, if $\|w_\varepsilon^2\|_{H^1(H_\varepsilon)} \to 0$, the proof is completed for $n = 2$.

Suppose now that $\|w_\varepsilon^2\|_{H^1(H_\varepsilon)} \nrightarrow 0$, $\|w_\varepsilon^2\|_{L^{p+1}(H_\varepsilon)} \nrightarrow 0$. In such case we can argue again as above.

Observe that we would finish after repeating the argument a finite number of times, concluding the proof.

The only possibility missing in our study is the following:

$$\text{at a certain } j, \quad \left\|w_\varepsilon^j\right\|_{H^1(H_\varepsilon)} \nrightarrow 0 \quad \text{and} \quad \left\|w_\varepsilon^j\right\|_{L^{p+1}(H_\varepsilon)} \to 0, \tag{41}$$

where $w_\varepsilon^j = u_\varepsilon - \sum_{k=1}^j u_k(\cdot - y_\varepsilon^k)$.

Step 4. The assertion (41) does not hold.

Suppose by contradiction (41). Let us define

$$H_\varepsilon^1 = \left\{x \in \mathbb{R}^N\colon\ \bar\lambda \cdot x \leqslant \frac{a_1}{\varepsilon}\right\},$$

where $\frac{\alpha_1}{2} < a_1 < \frac{2\alpha_1}{3}$. We claim that

$$\left\|w_\varepsilon^j\right\|_{L^{p+1}(H_\varepsilon^1)} \nrightarrow 0. \tag{42}$$

By (41) there exists $\delta > 0$ such that

$$\|u_\varepsilon\|^2_{H^1(H_\varepsilon)} \geqslant \sum_{k=1}^{j} \big\|u_k\big(\cdot - y_\varepsilon^k\big)\big\|^2_{H^1(H_\varepsilon)} + \delta. \tag{43}$$

Let us fix $R > 0$ large enough and choose a cut-off function ϕ satisfying the following:

$$\begin{cases} \phi = 0 & \text{in } \left(\bigcup_{k=1}^{j} B\big(y_\varepsilon^k, R\big)\right) \cup \big(H_\varepsilon^1\big)^c, \\ \phi = 1 & \text{in } H_\varepsilon \setminus \left(\bigcup_{k=1}^{j} B\big(y_\varepsilon^j, 2R\big)\right), \\ 0 \leqslant \phi \leqslant 1, \\ |\nabla \phi| \leqslant C/R. \end{cases}$$

We multiply (18) by ϕu_ε and integrate to obtain

$$\int_{\mathbb{R}^N} \phi |\nabla u_\varepsilon|^2 + u_\varepsilon(\nabla u_\varepsilon \cdot \nabla \phi) + V(\varepsilon x)\phi u_\varepsilon^2 = \int_{\mathbb{R}^N} g_\varepsilon(x, u_\varepsilon)\phi u_\varepsilon^2 + \bar{\lambda} \cdot h_\varepsilon(x)\phi u_\varepsilon^2.$$

Therefore, by using (fg) and the properties of the cut-off ϕ we get

$$\int_{H_\varepsilon \setminus (\bigcup_{k=1}^{j} B(y_\varepsilon^j, 2R))} \big(|\nabla u_\varepsilon|^2 + c u_\varepsilon^2\big) - \frac{C}{R} \leqslant C \int_{H_\varepsilon^1 \setminus (\bigcup_{k=1}^{j} B(y_\varepsilon^j, R))} u_\varepsilon^{p+1}. \tag{44}$$

Observe moreover that by regularity arguments $u_\varepsilon(\cdot + y_\varepsilon^k) \to u_k$ in H^1_{loc}. Then (43) implies that the left hand term in (44) is bounded from below: this finishes the proof of (42).

Then, we can repeat the whole procedure: there exists $y_\varepsilon^{j+1} \in H_\varepsilon^1$ such that $u_\varepsilon(\cdot + y_\varepsilon^{j+1}) \rightharpoonup u_{j+1}$. Define $w_\varepsilon^{j+1} = w_\varepsilon^j - u_{j+1}(\cdot - y_\varepsilon^{j+1})$. Observe that since $\|w_\varepsilon^j\|_{L^{p+1}(H_\varepsilon)} \to 0$, we have that $\operatorname{dist}(y_\varepsilon^{j+1}, H_\varepsilon) \to +\infty$.

Now we go on as above, replacing H_ε with H_ε^1. If for certain $j' \geqslant j+1$ we have

$$\big\|w_\varepsilon^{j'}\big\|_{H^1(H_\varepsilon^1)} \nrightarrow 0 \quad \text{and} \quad \big\|w_\varepsilon^{j'}\big\|_{L^{p+1}(H_\varepsilon^1)} \to 0,$$

we argue again as in the beginning of Step 2 to deduce that $\|w_\varepsilon^{j'}\|_{L^{p+1}(H_\varepsilon^2)} \nrightarrow 0$, where

$$H_\varepsilon^2 = \left\{x \in \mathbb{R}^N \colon \bar{\lambda} \cdot x \leqslant \frac{a_2}{\varepsilon}\right\},$$

with $a_1 < a_2 < \frac{2\alpha_1}{3}$.

In so doing we can again continue our argument, eventually introducing

$$H_\varepsilon^l = \left\{ x \in \mathbb{R}^N \colon \bar{\lambda} \cdot x \leqslant \frac{a_l}{\varepsilon} \right\},$$

with $a_{l-1} < a_l < \frac{2\alpha_1}{3}$.

Since all limit solutions u_k are bounded from below in norm, we finish after repeating the argument a finite number of times. Therefore, we obtain

$$\begin{aligned} &y_\varepsilon^k \in H_\varepsilon \quad \forall k = 1, \dots, j, \\ &\operatorname{dist}\big(y_\varepsilon^k, H_\varepsilon\big) \to \infty \quad \forall k = j+1, \dots, n, \\ &\left\| u_\varepsilon - \sum_{k=1}^n u_k\big(\cdot - y_\varepsilon^k\big) \right\|_{H^1(H_\varepsilon^q)} \to 0, \quad \text{for a suitable } q. \end{aligned}$$

This implies that

$$\| w_\varepsilon^j \|_{H^1(H_\varepsilon)} \leqslant \left\| u_\varepsilon - \sum_{k=1}^n u_k\big(\cdot - y_\varepsilon^k\big) \right\|_{H^1(H_\varepsilon)} + o_\varepsilon(1) = o_\varepsilon(1)$$

but this is in contradiction with $\| w_\varepsilon^j \|_{H^1(H_\varepsilon)} \nrightarrow 0$ assumed in (41).

A.3. Final remarks

Here we discuss some slight extensions of our result. As we shall see, a couple of hypotheses of Theorem 1.1 can be relaxed. However, we have preferred to keep Theorem 1.1 as it is, because in this form the proof becomes more direct and clear. So, let us now discuss those extensions of our results, as well as the modifications needed in the proofs.

1. *Condition* (f0). The C^1 regularity of $f(u)$ implies that all ground states of $(\mathcal{LP}_k)$ are radially symmetric (actually, $C^{0,1}$ regularity suffices). However, this is not really necessary in our arguments. Indeed, in [9] it is proved that the set $\mathcal{S}$ is compact, up to translations, even for continuous $f(u)$. So, in Section 3 it suffices to take $\gamma(t)$ related to $U \in \mathcal{S}$ such that

$$\int_{\mathbb{R}^N} U(x)x = 0.$$

Moreover, we cannot use compact embeddings of $H_r^1(\mathbb{R}^N)$ in the proof of Lemma 2.4: the proof of that lemma must be finished by making use of concentration-compactness arguments.

2. *Condition* (V0). The lower bound on V is strictly necessary in our arguments; the upper bound, though, has been imposed to make many computations have a simpler form. Indeed, condition (V0) can be replaced with

(V0′) $0 < \alpha_1 \leqslant V(x)$, $x \in \mathbb{R}^N$.

In such case, some technical work is in order. First, we need to consider the norm

$$\|u\|_V = \left(\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 \right)^{1/2},$$

and the Hilbert space H_V of functions $u \in H^1(\mathbb{R}^N)$ such that $\|u\|_V$ is finite. Then, it is not obvious that the solutions $U \in \mathcal{S}$ belong to H_V. Therefore, we need to define a cut-off function ψ_ε such that $\psi_\varepsilon = 1$ in B_2^ε, $\psi_\varepsilon = 0$ in $(B_3^\varepsilon)^c$ and $|\nabla \psi_\varepsilon| \leqslant C\varepsilon$.

The cone $\mathcal{C}_\varepsilon$ defined in (13) is to be replaced with

$$\overline{\mathcal{C}_\varepsilon} = \left\{ \psi_\varepsilon \gamma_t(\cdot - \xi)\colon\ t \in [0, 1],\ \xi \in \overline{B_0^\varepsilon} \cap E \right\}.$$

The estimates that would follow become more technical, but no new ideas are required.

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